4 - Elementary singularities

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Classification of elementary singularities

A vector field X in $(\mathbb{C}^d, 0)$ is elementary if its linear part has at least a non-zero eigenvalue. For $d=2$, we get λ_1, λ_2 eigenvalues, $\lambda_1 \neq 0$. $\lambda := \frac{\lambda_2}{\lambda_1}$.

Def: An elementary vector field X in $(\mathbb{C}^2, 0)$ is

- in the POINCARÉ domain if $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$
- in the (strict) SIEGEL domain if $\lambda \in \mathbb{R}_{<0}$] large SIEGEL domain $\lambda \in \mathbb{R}_{\leq 0}$
- degenerate (or saddle-node) if $\lambda = 0$

Results:

1) If X is Poincaré, $X \overset{hol}{\cong} \lambda_1 x \partial_x + (\lambda_2 y + \varepsilon x^n) \partial_y$, $(\lambda_2 - n\lambda_1)\varepsilon = 0$ $n \in \mathbb{N}^*$
(POINCARÉ-DULAC) $\qquad$ ↑ resonances

2) If X is Siegel, X admits exactly two (smooth transverse) convergent separatrices.
- MATTEI-MOUSSU 1980: the holonomy of any of the separatrices determines the equivalence class of the induced foliation $\mathcal{F}$
  $(h(z) = e^{2\pi i \lambda} z(1 + o(1)))$.
- PEREZ-MARCO, YOCCOZ 1994: Any (conjugacy class of) germ $h: (\mathbb{C}, 0) \circlearrowleft$ with $h'(0) = e^{2\pi i \lambda}$ $\lambda \in \mathbb{R}_{<0}$ can be realized as the holonomy of a foliation in the (strict) Siegel domain (with invariant λ)

3) If X is elementary degenerate, then X admits exactly two separatrices:
- the strong one, tangent to the λ_1 eigenspace, is always convergent.
- the weak one, tangent to the 0 eigenspace, might be divergent ($\sim$ central manifold)

Ex: EULER vector field $X = x^2 \partial_x + (y - x)\partial_y$ $\{x = 0\}$ strong, $\{y = \sum_{n=1}^{\infty} (n-1)! x^n\}$ weak

MARTINET-RAMIS (1982): The holonomy (tangent to the identity) of the strong separatrix determines the equivalence class of $\mathcal{F}$. All tangent to the identity germs (with affine modulus) can be realised as the holonomy of a saddle-node.

Formal classification [ILY.-YAK. §4]

$z = (x^1, \dots, x^d)$ notation: $\sum_{j=1}^{d} \chi^{j,(n)} \partial_{x^j}$ $\chi^j = \sum_n \chi^{j,(n)}$

$$\chi = \sum_{n\geq 1} \chi^{(n)} \partial_z$$ $\chi^{(n)}\partial_z$ homogeneous vector field of degree n

We say that χ and $\tilde{\chi}$ are (analitically/formally) conjugate, and write $\chi \overset{hol/for}{\cong} \tilde{\chi}$, if $\exists\, \Phi$ diffeomorphism s.t. $\Phi_* \chi = \tilde{\chi}$ ($\Leftrightarrow d\Phi \circ \chi = \tilde{\chi} \circ \Phi$ (☆))

$$\Phi_*\chi := d\Phi \circ \chi \circ \Phi^{-1}$$

First, take $\Phi = \Phi^{(1)}$ to be linear. The conjugacy equation gives $\Phi^{(1)} \circ \chi^{(1)} = \tilde{\chi}^{(1)} \circ \Phi^{(1)}$:
We may assume that $\chi^{(1)}$ is in (lower triangular) Jordan normal form.

Next, we consider $\Phi = \mathrm{id} + \Phi^{(m)}$, $m \geq 2$ (in particular, $\Phi^{(1)} = \mathrm{id}$)
We study (☆):

$$d\Phi \circ \chi = (I + d\Phi^{(m)})\chi = \sum_n \overset{n}{\chi^{(n)}\partial_z} + \sum_n \overset{m-1+n}{d\Phi^{(m)}\chi^{(n)}\partial_z}$$

$$\tilde{\chi} \circ \Phi = \sum \tilde{\chi}^{(n)}\left(z + \Phi^{(m)}(z)\right)\partial_z \overset{\text{Taylor}}{=} \sum_n \overset{n}{\tilde{\chi}^{(n)}\partial_z} + \overset{n-1+m}{d\tilde{\chi}^{(n)}\Phi^{(m)}\partial_z} + \overset{n-2+2m}{\tfrac{1}{2} d^2\tilde{\chi}^{(n)}(\Phi^{(m)},\Phi^{(m)})\partial_z} + \dots$$

We check the homogeneous parts of (☆) up to order m.

$h=1$: $\chi^{(1)} = \tilde{\chi}^{(1)}$
$\vdots$
$h=m-1$: $\chi^{(m-1)} = \tilde{\chi}^{(m-1)}$

$h=m$: $\chi^{(m)} + d\Phi^{(m)}\chi^{(1)} = \tilde{\chi}^{(m)} + d\tilde{\chi}^{(1)}\Phi^{(m)}$ (☆$_m$)

We can rewrite (☆$_m$) as: $\chi^{(m)} = \tilde{\chi}^{(m)} - [\underbrace{A\partial_z}_{=\chi^{(1)}}, \Phi^{(m)}\partial_z]$ ← LIE bracket

$\Phi^{(m)}\partial_z$: vector field, truncation of infinitesimal generator of Φ.

We proceed recursively on m: we want to solve the linear system $(\star_m)$, where:

$\chi^{(m)}$ are given by the problem

$\Phi^{(m)}\partial_z$ and $\tilde{\chi}^{(m)}$ are the unknown, with $\tilde{\chi}^{(m)}$ to be taken as simple as possible

Set $\mathcal{L}_A : \mathcal{D}^{(m)} \to \mathcal{D}^{(m)}$. $(\star_m)$ becomes $\chi^{(m)} = \tilde{\chi}^{(m)} - \mathcal{L}_A(\Phi^{(m)}\partial_z)$

$\eta \mapsto \mathcal{L}_A \eta = [A\partial_z, \eta]$

$\mathcal{D}^{(m)}$ = Vector space of homogeneous vector fields, basis: $\overbrace{\left(\eta_k^I = z^I \partial_k,\ k=1\dots d,\ |I| = m\right)}^{\mathcal{B}^{(m)}}$

Let us compute $\mathcal{L}_A$ on η_k^I.

Suppose first that $A = \mathrm{Diag}(\alpha_1 \dots \alpha_d)$. Then $A\partial_z = \sum \alpha_j x^j \partial_j$, and.

$$\mathcal{L}_A(\eta_k^I) = \sum_{j=1}^{d} \alpha_j x^j \partial_j (x^I \partial_k) - x^I \partial_k \left(\sum_{j=1}^{d} \alpha_j x^j \partial_j \right) =$$

$$= \sum_{j=1}^{d} \alpha_j x^j i_j \cdot x^{I-e_j} \partial_k - x^I \alpha_k \partial_k = \left(\langle \alpha, I \rangle - \alpha_k \right) \eta_k^I$$

In this case, the matrix $L^{(m)}$ representing $\mathcal{L}_A$ in the basis $\mathcal{B}^{(m)}$ is diagonal, and

$\mathrm{Ker}\, L^{(m)} = \mathrm{Vect}\left(\eta_k^I,\ \langle \alpha, I \rangle - \alpha_k = 0 \right)$.

Def: $\mathbb{C}\eta_k^I$ s.t. $\langle \alpha, I \rangle - \alpha_k = 0$ is called a resonant monomial.

If A has a non-semisimple part, we are led to work with $\overset{\alpha_{j-1}}{\overset{\shortparallel}{\alpha_j}} x^{j-1} \partial_j =: \xi$

and $\mathcal{L}_\xi(\eta_k^I) = \alpha_j x^{j-1} i_j z^{I-e_j} \partial_k - \underbrace{z^I \alpha_j \partial_j}_{\text{if } k=j-1} = \alpha_j i_j \underset{e_{j-1} > e_j}{x^{I+e_{j-1}-e_j}} \partial_k - \alpha_j z^I \underset{\overset{\vee}{k}}{\partial_{k+1}}$

If we order (k, I) in lexicographic order, have higher order than (k, I), and $L^{(m)}$ is (upper) triangular, with $\langle \alpha, I \rangle - \alpha_k$ on the diagonal.

We deduce that $\mathcal{D}^{(m)} = V^{(m)} \oplus R^{(m)}$, with $V^{(m)} = \mathcal{L}_A(\mathcal{D}^{(m)})$, $R^{(m)} = \mathrm{Vect}(\eta_k^I, \langle \alpha, I \rangle - \alpha_k = 0)$

Can write $\chi^{(m)} = -\mathcal{L}_A(\underset{\mathcal{D}^{(m)}}{\eta}) + \underset{\tilde{\chi}^{(m)}}{\rho} \in R^{(m)}$

By arguing by induction, we get:

Thm (Poincaré-Dulac) $\forall \mathcal{X}$ is formally conjugate to its P-D. normal form: $\tilde{\mathcal{X}} = \mathcal{X}^{(1)} + \mathcal{X}^{res}$, where $\mathcal{X}^{(1)}$ is in lower Jordan form, and $\mathcal{X}^{res}$ has only resonant monomials.

Rem: 1) We can get the PD normal form up to any high order by a polynomial change of coordinates, but in general the conjugacy might diverge
2) The PD normal form is not unique, since the choice of $\Phi^{(m)}$ changes the datum of $\mathcal{X}^{(n)}$ $n>m$ in the next steps. The choice of $\Phi^{(m)}$ is not unique in presence of resonances.

Formal classification 2D

. Rem: $\alpha_1 i_1 + \alpha_2 i_2 = \alpha_1 \iff \alpha_1 (\underbrace{i_1 - 1}_{\neq 0}) + \alpha_2 \underbrace{i_2}_{\neq 0} = 0 \iff \frac{\alpha_2}{\alpha_1} = -\frac{i_1 - 1}{i_2} \in \mathbb{Q}$

similarly for $\alpha_1 i_1 + \alpha_2 i_2 = \alpha_2 \to \frac{\alpha_2}{\alpha_1} = -\frac{i_1}{i_2 - 1}$

We deduce that if $\alpha \notin \mathbb{Q}$, $\mathcal{X} \overset{for}{\cong} \alpha_1 x \partial_x + \alpha_2 y \partial_y$

If $\alpha \in \mathbb{Q}_{>0}$: we must have $i_1 = 0$, $i_2 = \frac{1}{\alpha} \in \mathbb{N}^* \to$ resonance $y^u \partial_x$ $\alpha_1 = u\alpha_2$

$i_2 = 0$, $i_1 = \alpha \in \mathbb{N}^* \to$ resonance $x^u \partial_y$ $\alpha_2 = u\alpha_1$

If $\alpha = -\frac{p}{q}$. $\to \frac{i_1 - 1}{p} = \frac{i_2}{q}$ for ∂_x, $\frac{i_1}{p} = \frac{i_2 - 1}{q}$ for $\partial_y \to$ resonances $(x^p y^q)^n (x\partial_x + y\partial_y)$

If $\alpha = 0$: $\alpha_1 (i_1 - 1) = 0$ for ∂_x, $\alpha_1 i_1 = 0$ for ∂_y. $\to$ resonances $y^n (x\partial_x + y\partial_y)$
$i_1 = 1$

Hence: Poincaré: $\alpha \in \mathbb{C}^* \setminus (\mathbb{R}_{\leq 0} \cup \mathbb{N}^* \cup \frac{1}{\mathbb{N}^*})$, $\mathcal{X} \overset{for}{\cong} \alpha_1 x \partial_x + \alpha_2 y \partial_y$

$\alpha = u$ or $\frac{1}{u}$, $u \in \mathbb{N}^*$, $\mathcal{X} \overset{for}{\cong} \alpha_1 x \partial_x + (u\alpha_1 y + \underset{\mathbb{C}}{\underset{\cap}{\varepsilon}} x^u) \partial_y$

Siegel: $\alpha \in \mathbb{R}_{<0} \setminus \mathbb{Q}$, $\mathcal{X} \overset{for}{\cong} \alpha_1 x \partial_x + \alpha_2 y \partial_y$

$\alpha = -\frac{p}{q} \in \mathbb{Q}_{<0}$, $\mathcal{X} \cong \alpha_1(x^p y^q) x \partial_x + \alpha_2 (x^p y^q) y \partial_y$.

$\alpha_1(0) = \alpha_1$, $\alpha_2(0) = \alpha_2$, both $\neq 0$.

Saddle-node: $\alpha=0$, $\quad X \overset{\text{for}}{\cong} x\, a(y)\,\partial_y + y^{\mu} b(y)\,\partial_y \quad a(0), b(0)\neq 0, \mu\geq 2.$

can be improved both by conjugacy and by equivalence.

Analytic classification [Ily-Yak §5]

Suppose for simplicity that $\alpha \in \mathbb{C}^d$ is non-resonant.

In the Poincaré-Dulac process, we are led to consider $\eta = \mathcal{L}_A^{-1}(-X^{(m)})$

In general, we invert the diagonal square submatrix of $L^{(m)}$ associated to the $\bigoplus E_\beta \leftarrow$ characteristic space.

$\beta = \langle\alpha, I\rangle - \alpha_k \neq 0$

Hence we are led to divide by $\beta_k^I := \langle \alpha, I\rangle - \alpha_k$, where $k=1\ldots d$, $|I| = m$.

When m grows β_k^I might be very small (even in the non-resonant case).

(small divisor problem).

This happens exactly when 0 is a cluster point of $\{\beta_k^I,\ k=1\ldots d,\ I\in\mathbb{N}^d\}$, or equivalently when $0 \in \mathrm{Conv\,Hull}(\alpha_1 \ldots \alpha_d) =: \Delta(\alpha)$

Def: $\alpha\in\mathbb{C}^d\setminus\{0\}$ (or X elementary vector field) is said to be:

- in the Poincaré domain if $0\notin\Delta(\alpha)$
- in the Large Siegel domain if $0\in\Delta(\alpha)$
- in the strict Siegel domain if $0\in\mathring{\Delta}(\alpha)$.

Rem: in 2D, this corresponds to: $\frac{\alpha_2}{\alpha_1 \neq 0}\in\mathbb{C}\setminus\mathbb{R}_{\leq 0}$, $\frac{\alpha_2}{\alpha_1}\in\mathbb{R}_{\leq 0}$, $\frac{\alpha_2}{\alpha_1}\in\mathbb{R}_{<0}$ respectively.

Poincaré domain [Ily-Yak §5]

Theorem (Poincaré-Dulac) Let X be in the Poincaré domain.

Then the number of resonant monomials is finite, and X is analytically conjugate to its P.D normal form.

Proof: There exists a complex number δ s.t. $\delta\Delta(\alpha)$ is contained in $\{\mathrm{Re}\,\alpha \geq 1\}$

Let $I \in \mathbb{N}^d$. $\langle \alpha, I \rangle = \sum \alpha_j i_j$, and $\mathrm{Re}(\delta \langle \alpha, I \rangle) \geq |I|$

In particular, if $|I| > \max\{\mathrm{Re}\,\delta\alpha_k\}$, the $\beta_k^I \neq 0$ and η_k^I is not resonant

To prove the convergence of the PD conjugacy, we use the classical tool of majorant series. (to reduce the problem to a fixed point theorem).

We write $\mathcal{X} = \mathcal{X}_D + \mathcal{X}_R + \mathcal{X}_T$ where $\mathcal{X}_D$ is diagonal (or Jordan), $\mathcal{X}_R$ is the resonant part, $\mathcal{X}_T$ is the tail part (may assume $T \in \mathfrak{M}^{m+1}\mathcal{D}$, $m >> 0$).

We want to solve the conjugacy equation $\Psi_* \underbrace{\tilde{\mathcal{X}}}_{= \mathcal{X}_D + \mathcal{X}_R} = \mathcal{X}$, which gives:

$$d\Psi \cdot (\mathcal{X}_D + \mathcal{X}_R) = (\mathcal{X}_D + \mathcal{X}_R + \mathcal{X}_T) \circ \Psi. \quad (\star_2)$$

Rem: we reversed the conjugacy equation $(\star_1)$, setting $\Psi = \Phi^{-1}$.

Write $\Psi = (id + \xi)$, with ξ of high order ($\geq m+1$).

Equation $(\star_2)$ becomes:

$$\cancel{\mathcal{X}_D} + \mathcal{X}_R + d\xi \cdot \mathcal{X}_D + d\xi \cdot \mathcal{X}_R = \cancel{\mathcal{X}_D} + \mathcal{X}_D \circ \xi + \mathcal{X}_R \circ \Psi + \mathcal{X}_T \circ \Psi$$

$$\mathcal{L}_D(\xi) = \underbrace{\mathcal{X}_T(id + \xi)}_{S = S_T(\xi)} + \underbrace{\mathcal{X}_R(id + \xi) - \mathcal{X}_R}_{V = V_R(\xi)} \underbrace{- d\xi \cdot \mathcal{X}_R}_{J = J_R(\xi)}$$

Finding a solution of $(\star_2)$ becomes finding a fixed point ξ for the operator $\mathcal{L}_D^{-1}(S_T + V_R + J_R)$, on a suitable functional space.

Majorant series [Ily-Yak §5]

The majorant operator is defined as $M : \mathbb{C}[[z]] \to \mathbb{R}[[z]]$

$\sum a_I z^I \prec \sum b_I z^I \overset{def}{\iff} |a_I| \leq |b_I| \;\forall I.$ $\quad \phi = \sum a_I z^I \mapsto \sum |a_I| z^I$

$\forall \rho > 0$, the majorant ρ-norm is $[\![\phi]\!]_\rho = \sup_{|z| \leq \rho} |M\phi(z)| = \underbrace{M\phi(\rho, \dots, \rho)}_{= N(\phi)(\rho)} \leq +\infty$.

$B_\rho := \{\phi \in \mathbb{C}[[z]] \mid [\![\phi]\!]_\rho < +\infty\}$

Can extend to $\mathbb{C}[[z]]^d$ ($[\![\Phi]\!]_\rho = [\![\phi^1]\!]_\rho + \dots + [\![\phi^d]\!]_\rho$) $\rightsquigarrow B_\rho^d$, and to vector fields: $\mathcal{D}_\rho = B_\rho^d \partial_z$

Prop $(B_\rho, \|\cdot\|_\rho)$ is complete

$(B_\rho, \|\cdot\|_\rho) \cong \left((a_I)_I \text{ s.t. } \sum |a_I| \rho^{|I|} < +\infty \right)$ isomorphic to $\ell^1(\mathbb{C})$, which is complete.

Rem: $B_\rho \subsetneq \mathcal{A}_\rho = \{ \phi : (\mathbb{D}_\rho)^d \to \mathbb{C}$ holomorphic, continuous on $\overline{\mathbb{D}}_\rho^d$, with $\|\phi\|_\infty \} \subsetneq B_{\rho'}$ $\forall \rho' < \rho$

Properties: $\|\phi\psi\|_\rho \leq \|\phi\|_\rho \cdot \|\psi\|_\rho$

$$\|\Phi \circ \Psi\|_\rho \leq \|\Phi\|_\sigma \qquad \sigma = \|\Psi\|_\rho$$

Lemma: let $\alpha = (\alpha_1 \ldots \alpha_d)$ be of Poincaré type.

Let m be big enough so that $\beta_k^I := \langle \alpha, I \rangle - \alpha_k \neq 0$ $\forall I$, $|I| > m$, $\forall k = 1 \ldots d$

Then $\mathcal{L}_D : \mathcal{M}^{m+1}\mathcal{D} \to \mathcal{M}^{m+1}\mathcal{D}$ is invertible, and $\|\mathcal{L}_D^{-1}\|_\rho \leq \left(\inf_{\substack{|I| > m \\ k = 1 \ldots d}} |\beta_{k,I}| \right)^{-1} < +\infty$

Proof. $\mathcal{L}_D$ sends $\eta_k^I = z^I \partial_k$ on $\beta_k^I \cdot \eta_k^I$. ($\Rightarrow \mathcal{L}_A$ is invertible).

Hence $\mathcal{L}_D \left(\sum_{k,I} a_k^I \eta_k^I \right) = \sum_{k,I} \beta_k^I a_k^I \eta_k^I$.

In the POINCARÉ domain, $|\beta_k^I|$ is bounded below by $\varepsilon > 0$. We deduce

$\|\mathcal{L}^{-1}(\eta)\|_\rho = \left\| \sum (\beta_k^I)^{-1} a_k^I \eta_k^I \right\|_\rho \leq \varepsilon^{-1} \|\eta\|_\rho$. $\square$

Rem: 1) $\mathcal{L}_A$ is unbounded 2) if A is non-diagonalizable, the estimate is similar: $\mathcal{L}_A$ is still invertible, but not necessarily diagonalizable, and the estimate is a little harder. 3) Working with $\mathcal{A}_\rho$ wouldn't work in general

Lemma: • $\|S(0)\|_\rho = O(\rho^{m+1})$ $\qquad (m+1 = \text{ord } T)$

$S(0) = N(S(0))(\rho) \in \mathcal{M}^{m+1}$

• S is Lipschitz on $B_\rho(\rho) := \{ \xi \mid \|\xi\|_\rho \leq \rho \}$, with Lipschitz constant $O(\rho^m)$.

Proof: $S(0) = \chi_T$, and χ_T has order $\geq m+1$, hence $\|S_T(0)\|_\rho = O(\rho^{m+1})$.

Let now $\xi_0, \xi_1 \in B_\rho(\rho)$.

Then $S(\xi_1) - S(\xi_0) = \chi_T(\mathrm{id} + \xi_1) - \chi_T(\mathrm{id} + \xi_0) = \int_0^1 d\chi_T\left(z + \tau\xi_1 + (1-\tau)\xi_0\right) \cdot (\xi_1 - \xi_0)\, d\tau$

$\|d\chi_T(z + \tau\xi_1 + (1-\tau)\xi_0) \cdot (\xi_1 - \xi_0)\|_\rho \leq \|d\chi_T\|_\sigma \cdot \|\xi_1 - \xi_0\|_\rho$, $\sigma = \|\mathrm{id} + \tau\xi_1 + (1-\tau)\xi_0\|_\rho$

$$\sigma \le \underset{\overset{\wedge\!\!|}{\rho}}{\|z\|_\rho} + \max\Big\{ \underset{\overset{\wedge\!\!|}{d\rho}}{\|\rho_1\|_\rho}, \underset{\overset{\wedge\!\!|}{d\rho}}{\|\rho_0\|_\rho} \Big\} \le (d+1)\rho.$$

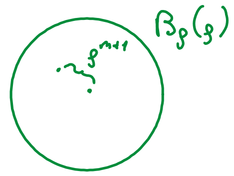
dX_T has order at least m, and $\|dX_T\|_\sigma = O(\sigma^m) = O\big((d+1)^m \cdot \rho^m\big)$

$\Rightarrow S$ is $O(\rho^m)$-Lipschitz. $\square$

Rem: $S : B_\rho(\rho) \subseteq B_\rho(\rho)$ for $\rho << 1$: $(m \geq 1)$.

The center is sent to $O(\rho^{m+1}) << \rho$.

The diameter of $S(B_\rho(\rho))$ is $\le C \cdot 2\rho \cdot \rho^m << \rho$.



Proof of the theorem (non-resonant case): Apply the Banach fixed point theorem to $\mathcal{L}_D^{-1} \circ S_T$, $m=1$, ρ s.t. $\varepsilon^{-1} \cdot O(\rho) << 1$. $\square$

General case: need to estimate V_R and J_R. Suppose $X^{(1)}$ diagonal.

- $V_R = S_R - S_R(0) \Rightarrow V_R$ is $O(\rho)$-Lipschitz, and V_R leaves $B_\rho(\rho)$ invariant for $\rho << 1$.
- J_R is in general unbounded. Goal: $\|\mathcal{L}_D^{-1} J_R\|_\rho = O(\rho)$

If $X^{(1)}$ is not diagonalisable: either estimate $\|\mathcal{L}_A^{-1}\|$, or $\|V_R\|_\rho$, $\|J_R\|_\rho$, with R which has a non-trivial (nilpotent) linear part.

Geometry of Poincaré foliations in 2D

- $X \cong \alpha_1 x \partial_x + \alpha_2 y \partial_y$ $\quad \alpha = \frac{\alpha_2}{\alpha_1} \in \mathbb{C} \setminus \mathbb{R}_{\le 0}$ (most remarks hold for $\alpha \in \mathbb{C}^*$)

(Multivalued) first integral: $s = x^{-\alpha} y \rightsquigarrow$ leaves $y = c\, x^\alpha$.

Separatrices: $\{x=0\}$, $\{y=0\}$, $\{y^q = c \cdot x^p\}$ if $\alpha = \frac{p}{q} \in \mathbb{Q}_{>0}$ (dicritical case).

- $X \cong \alpha_1 x \partial_x + (u\alpha_1 y + \varepsilon x^u)\partial_y$

(Multivalued) first integral: $s = x^{-u} y - \varepsilon \log x \rightsquigarrow$ leaves $y = (c + \varepsilon \log x)\, x^u$.

Only one separatrix $\{x=0\}$.

Siegel and degenerate singularities : separatrices

Spec $X^{(1)} = \{\alpha_1, \alpha_2\}$, $\alpha_1 \neq 0$, $\alpha_2 \in \mathbb{R}_{\leq 0}\alpha_1$. $\alpha = \frac{\alpha_2}{\alpha_1} \in \mathbb{R}_{\leq 0}$

If $\alpha \in \mathbb{R}_{<0}$: we have two separatrices. (known since Briot-Bouquet, 1856)

If $\alpha = 0$, we get a separatrix tangent to the α_1 eigenspace

These results are a consequence of Hadamard-Perron theorem (1901-1928)

Thm: [Ily-Yak Thm 7.1, Kat-Has Thm 6.2.8 for maps] Let X be a vector field on $(\mathbb{C}^d, 0)$

Let H be a open half plane in $\mathbb{C}$, and let E_H be the sum of generalised eigenspaces associated to $\alpha \in H$. of $X^{(1)}$.

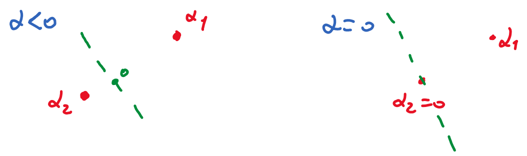
Suppose that $0 \notin H$. Then $\exists V_H$ invariant manifold for X, tangent to E_H.

Idea: up to (orbital) equivalence, may assume $H = \{\mathrm{Re}\,\alpha < \delta\}$.

Then $f = \exp(X)$ has eigenvalues e^{α}, α eigenvalues of $X^{(1)}$, and $\exp(H) = \{\lambda \mid |\lambda| < e^{\delta}\}$

If $\delta = 0$, V_H is the stable manifold for f. (Perron version for $\delta \neq 0$).

We apply Hadamard-Perron :



Up to holomorphic change of coordinates, may assume that the stable manifolds are the coordinate axes.

$\alpha < 0$: $X \cong \alpha_1 x\,(1 + o(1))\,\partial_x + \alpha_2 y\,(1 + o(1))\,\partial_y$.

$\alpha = 0$: $X \cong f(x,y)\,\partial_x + y\,(\dots)$

Classification up to formal equivalence

We recall the P-D formal normal forms of vector fields in the large Siegel domain

$\alpha < 0$: $X \overset{for}{\cong} x\,\alpha_1(u)\,\partial_x + y\,\alpha_2(u)\,\partial_y$, $\alpha = \frac{\alpha_2}{\alpha_1}$

where α_1, α_2 constant in the non-resonant case, and

$\alpha_1, \alpha_2 \in \mathbb{C}[[u]]$, $u = x^p y^q$, $\alpha_j(0) = \alpha_j$ if $\alpha = -\frac{p}{q} \in \mathbb{Q}_{<0}$.

$\alpha = 0$: $X \overset{for}{\cong} x\, a(y)\partial_x + y^\mu b(y)\partial_y \qquad a(0) = \alpha_1,\ b(0) \neq 0,\ \mu \geq 2.$

Rem: in these coordinates, $y = 0$ is the strong separatrix, and $x = 0$ gives the weak separatrix (tangent to the 0 eigenvalue, i.e. the "central manifold").

Siegel case: can divide by α_1, and get: $X \overset{for}{\sim} x\partial_x + y\,\alpha(u)\partial_y$.

In the non-resonant case, $\alpha(u) = \alpha$, and X is in formal normal form.

In the resonant case, we perform a change of coordinates $\Phi(x,y) = (x, y\varphi(u))$ with $\varphi \in \mathbb{C}[[u]]^\times$. We get the normal form:

$$X \overset{for}{\sim} X^F := x\partial_x + y\,\alpha(u)\partial_y, \qquad \alpha(u) = -\frac{p}{q} + u^r + \beta u^{2r} \quad \beta \in \mathbb{C}.$$

Rem: can replace $\alpha(u)$ by $-\frac{p}{q} + \frac{u^r}{1-\beta u^r}$, as in [Il-Ya], up to a mistake there. The two formal forms should be holomorphically conjugate, to check.

Degenerate case: Can divide by $a(y)$, and get $X \overset{for}{\sim} x\partial_x + y^\mu c(y)\partial_y \quad c(0) \neq 0.$

Up to a change of coordinates of the form $\Phi(x,y) = (x, y\varphi(y))$, $\varphi \in \mathbb{C}[[y]]^\times$, we get

$X \overset{for}{\sim} x\partial_x + y^{r+1}(1+\beta y^r)\partial_y$, $\beta \in \mathbb{C}$, $\mu = r+1$, $r \geq 1$. Set $X^F = x^{r+1}(1+\beta x^r)\partial_x + y\partial_y$.

Rem: can replace by $\frac{y^{r+1}}{1-\beta y^r}\partial_y$, this is conjugate to $y^{r+1}(1+\beta y^r)\partial_y$, by setting $\tilde{y} = \frac{y}{\sqrt[r]{1-\beta y^r}}$.

Notice also that $X^F \sim x^{r+1}\partial_x + \frac{y}{1+\beta x^r}\partial_y \cong x^{r+1}\partial_x + y(1-\beta x^r)\partial_y$

High tangency to X^F along the separatrices

Lemma: let X^F be the orbital formal normal form described above.

For any $N \in \mathbb{N}$, there is $R_N \in \mathbb{C}\{x,y\}$ such that:

$X \sim X^F + x^N y^N R_N \partial_y \qquad (\alpha < 0)$

$X \sim X^F + x^N R_N \partial_y \qquad (\alpha = 0)$

Rem: $\{xy = 0\}$ and $\{x = 0\}$ are the separatrices of X in the coordinates above

This statement can be found in [Il-Ya, Lemmas 22.3 and 22.15] A similar statement with $X \cong \alpha_1 x(1 + xy\,\varepsilon_1)\partial_x + \alpha_2 y(1 + xy\,\varepsilon_2)\partial_y \quad \varepsilon_1, \varepsilon_2 \in \mathbb{C}\{x,y\}$ can be found in [Mat-Mou, Appendix 2], based on majorant series techniques.

Proof (Siegel case). Firstly, we may assume that the separatrices are $\{xy=0\}$.

By Poincaré-Dulac (and similar arguments), we may also assume that X is in the formal orbital form up to high order:

$X = X^F + R\partial_y$ with $R \in \mathfrak{M}^M$, $y|R$, $M \gg 0$ (to be determined as a function of N)

In particular, we get $R \in \langle x^I y^J\rangle \cap \mathfrak{M}^M$, $I=0$, $J=1$. We proceed by induction to get $I=J=N$.

Suppose we have $R \in \langle x^I y^J\rangle$ $I,J \in \mathbb{N}$.

$\Phi = \Psi^{-1}$

We perform a change of coordinates $\Psi(x,y) = (x, y + y^J\varphi(x))$, $\varphi \in \langle x^I\rangle$, in order to conjugate X with $\tilde{X} = X^F + \tilde{R}\partial_y$, with $\tilde{R} \in \langle x^I y^{J+1}\rangle$. We get

$$\Phi^* X = x\partial_x + \frac{1}{1+Jy^{J-1}\varphi}\left[-y^J x\varphi'(x) + (y + y^J\varphi(x))\,\alpha(u\circ\Psi) + R\circ\Psi\right]\partial_y$$

$$R = y^J R_0(x) + \langle y^{J+1}\rangle. \qquad R_0 \in \langle x^I\rangle$$

We check the y^J-jet of $\Phi^* X - X^F$, and get: $\left(-x\varphi'(x) + \alpha\varphi(x) - J\alpha\varphi + R_0(x)\right)\partial_y$.

We want to solve $x\varphi' + \alpha(J-1)\varphi = R_0(x)$ for φ.

This can be done explicitly: $\varphi = \sum \varphi_n x^n$, $R_0(x) = \sum r_n x^n$ $\rightsquigarrow$ $n\varphi_n + \alpha(J-1)\varphi_n = r_n$.

$\rightsquigarrow$ $\varphi_n = \dfrac{r_n}{n+\alpha(J-1)}$. If α is non-resonant, $n + \alpha(J-1) \neq 0 \ \forall n$, and it is easy to check that φ is convergent since R_0 is.

If α is resonant, we might have $n + \alpha(J-1) = 0$. But by assumption R_0 has order $\geq M - J \geq M - N$. If $n = -\underset{}{\overset{\in \mathbb{Q}_{<0}}{\alpha}}(J-1)$ $\Rightarrow$ $n \leq -\alpha(N-1)$

Take M s.t. $-\alpha(N-1) \leq M - N$, i.e. $M \geq N - \alpha(N-1)$, and we have no resonance issues occurring in our equation.

To go from $R \in \langle x^I y^J\rangle$ to $R \in \langle x^{I+1} y^J\rangle$ we proceed analogously, by setting $\Psi(x,y) = (x, y + x^I\psi(y))$. We get:

$$\Phi^* X = x\partial_x + \frac{1}{1 + x^I\psi'}\left(-Ix^I\psi + (y + x^I\psi(y))\,\alpha(u\circ\Psi) + R\circ\Psi\right)\partial_y$$

By checking the x^I-jet of $\Phi^* X - X^F$, we get: $-I\psi + \alpha\psi + R_0 - \psi' y\alpha$

As before we want to solve $\alpha y\psi' + (I-\alpha)\psi = R_0$, which gives

$\alpha n\psi_n + (I-\alpha)\psi_n = r_n$ $\rightsquigarrow$ $\psi_n = \dfrac{r_n}{\alpha(n-1)+I}$, and we conclude as we did above.

Rem: the degenerate case is analogous.

Holonomy along separatrices

Let $\mathcal{F}$ be a foliation given by $X = x\partial_x + \alpha y f(x,y)\partial_y$, with $f(0,0)=1$.
We may always assume this is the case when $\alpha \in \mathbb{R}_{\leq 0}$, and $C=\{y=0\}$ is a separatrix.
In order to compute the holonomy h with respect to a path $(\gamma,0)$ in C, we denote by $\Gamma(t,y):=(\gamma(t), h(t,y))$ the intersection of the leaf L_y at $(\gamma(0),y)$ with the transverse $x=\gamma(t)$ (for t small, and extend by finite coverings...)
By setting $\omega = x\,dy - \alpha y f\,dx$, we have that

$$\omega_{\Gamma(t,y)}\left(\frac{\partial}{\partial t}\Gamma(t,y)\right)=0.$$

We get

$$\gamma(t)\cdot\frac{\partial h}{\partial t}(t,y) - \alpha h(t,y)\cdot f(\Gamma(t,y))\cdot\gamma'(t)=0.$$

We apply to $\gamma = e^{2\pi i t}$, and $h=\sum_{k\geq 0} h_k y^k$, and get $\sum h_k' y^k = 2\pi i\alpha\, h(t,y)\cdot f\circ\Gamma(t,y)$,
with initial condition $h_k(0)=\begin{cases}1 & k=1\\ 0 & k\neq 1\end{cases}$.

Applying this formula to the orbital normal forms X^F, we get:

- Siegel non-resonant: $f\equiv 1$, and $h^F(y) = e^{2\pi i\alpha}y$.
- Siegel resonant: $\alpha=-\frac{p}{q}$ $f = 1+u^r+\beta u^{2r}$ $r\geq 1$, $\beta\in\mathbb{C}$, $u=x^py^q$.

$$h^F(y) = e^{-2\pi i\frac{p}{q}}y\left(1+ay^{qr}+a\left(\beta+\frac{qr+1}{2}a\right)y^{2qr}+o(y^{2qr})\right) \qquad a:=2\pi i\alpha = -2\pi i\frac{p}{q}.$$

$$\overset{\text{for}}{\cong} e^{-2\pi i\frac{p}{q}}y\left(1-y^{qr}+\left(\frac{\beta}{a}+\frac{qr+1}{2}\right)y^{2qr}\right)$$

- degenerate: $\alpha=0$, $X^F = x\partial_x + y^{r+1}(1+\beta y^r)\partial_y$, $\{y=0\}$ strong separatrix

$$h^F(y) = y\left(1+ay^r+a\left(\beta+\frac{r+1}{2}a\right)y^{2r}+o(y^{2r})\right)\overset{\text{for}}{\cong} y\left(1-y^r+\left(\frac{\beta}{a}+\frac{r+1}{2}\right)y^{2r}\right) \qquad a:=2\pi i$$

A direct computation yields $h_k' = -2\pi i\frac{p}{q}\left(h_k + e^{2\pi i p r t}\sum_{\substack{I\in\mathbb{N}^{qr+1}\\ |I|=k}} h_I + \beta e^{2\pi i 2prt}\sum_{\substack{J\in\mathbb{N}^{2qr+1}\\ |J|=k}} h_J\right)$

For $k=0$, we get $h_0' = -2\pi i \frac{p}{q} h_0$, $h_0(0)=0 \Rightarrow h_0 \equiv 0$.

For $k \le qr$, we get $h_k' = -2\pi i \frac{p}{q} h_k \Rightarrow h_k = e^{-2\pi i \frac{p}{q} t} h_k(0) = \begin{cases} 0 & k \neq 1 \\ e^{-2\pi i \frac{p}{q}} & k=1 \end{cases}$.

For $k = qr+1$, we get $h_k' = -2\pi i \frac{p}{q}\left(h_k + e^{2\pi i p r t} h_1^{qr+1}\right) = -2\pi i \frac{p}{q}\left(h_k + e^{-2\pi i \frac{p}{q} t}\right)$

$\Leftrightarrow h_{qr+1}(t) = -2\pi i \frac{p}{q} t \, e^{-2\pi i \frac{p}{q} t} = t h_1'(t)$

For $qr+1 < k < 2qr+1$, we get $h_k' = -2\pi i \frac{p}{q} h_k$, and again $h_k \equiv 0$.

For $k = 2qr+1$, we get $h_k' = -2\pi i \frac{p}{q}\left(h_k + (qr+1) e^{2\pi i p r t} h_1^{qr} h_{qr+1} + \beta e^{2\pi i 2 p r t} \cdot h_1^{2qr+1}\right)$

$= -2\pi i \frac{p}{q}\left(h_k - (qr+1) 2\pi i \frac{p}{q} t \, e^{-2\pi i \frac{p}{q} t} + \beta e^{-2\pi i \frac{p}{q} t}\right)$

$\Leftrightarrow h_{2qr+1}(t) = -2\pi i \frac{p}{q} t \left(\beta - \frac{qr+1}{2} \cdot 2\pi i \frac{p}{q} t\right) e^{-2\pi i \frac{p}{q} t}$

The degenerate case is analogous.

The lemma above implies that the holonomies h of F with respect to these separatrices are formally conjugate to the corresponding holonomies h^F.

Rem: in particular, any formal class of germ with multiplier $\lambda = e^{2\pi i \alpha}$ can be achieved by the holonomy of a foliation with index α, $\alpha \in \mathbb{R}_{\le 0}$ (w.r.t. the strong separatrix if $\alpha = 0$).

Our next goal is to achieve a similar result for analytic classes.

The holonomy determines the equivalence class (Siegel case)

Thm (Mattei-Moussu) $X = x\partial_x + \alpha y(1+f(x,y))\partial_y$ $\quad h$ holonomy w.r.t. $\{y=0\}$ $\quad f(0,0)=0$

$\tilde{X} = x\partial_x + \alpha y(1+\tilde{f}(x,y))\partial_y$ $\quad \tilde{h}$ " " " . $\quad \tilde{f}(0,0)=0$.

Then X and $\tilde{X}$ are orbitally equivalent $\Leftrightarrow$ h and $\tilde{h}$ are conjugate

Proof: $\Rightarrow$ by construction of holonomy.

($\Leftarrow$) Up to change of coordinates, we may assume:

- $X, \tilde{X}$ defined on $U \supset K = \overline{\mathbb{D}}_1^2$
- $x \mid f$ (Lemma), $|f| \leq \frac{1}{2}|x|$ on K, analogous for $\tilde{X}$.

Set $C_0 = \{y=0\}$ the separatrix. Let ϕ be a conjugacy between h and $\tilde{h}$, computed on the transversals $\{x=1\}$ with respect to $\gamma_0(t) = (e^{2\pi i t}, 0)$: $\phi \circ h = \tilde{h} \circ \phi$

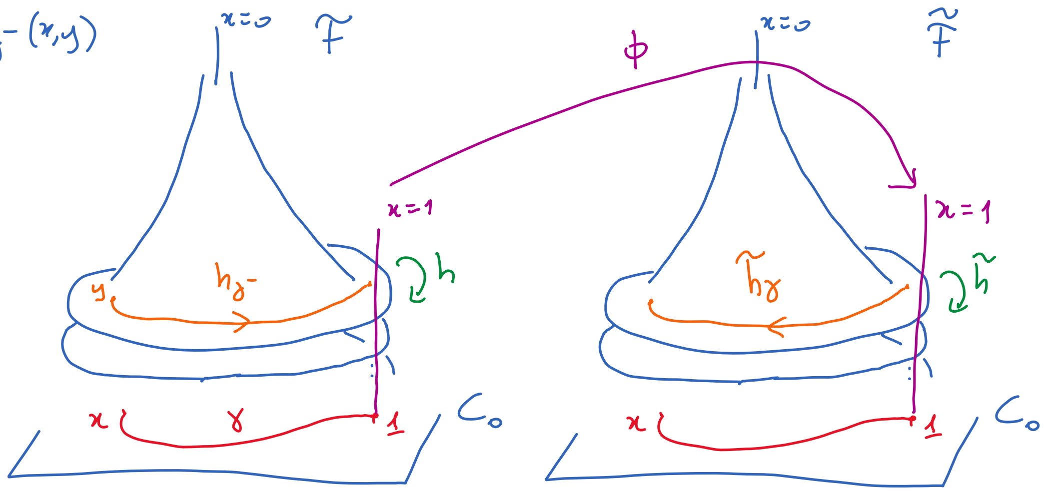
We extend ϕ via the foliations to $\mathbb{C}^2 \setminus \{x=0\}$, as follows.

For any $x \neq 0$, pick γ path in $\overline{\mathbb{D}}^* \times \{0\}$ from 1 to x, and define:

$\Phi(x,y) = \tilde{h}_\gamma \circ \phi \circ h_{\gamma^-}(x,y)$

h_γ = holonomy for $\mathcal{F}$

$\tilde{h}_\gamma$ = " " $\tilde{\mathcal{F}}$

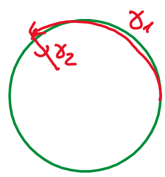


Properties:

1) Φ does not depend on the choice of γ (consequence of the conjugacy equation)

2) Φ is defined on a neighborhood of $0 \setminus \{x=0\}$.

3) Φ is bounded around $\{x=0\}$ $\Rightarrow$ it extends to a biholomorphism, which preserves the leaves of the foliations by construction.



2 & 3: via estimates, using γ of the form $\gamma = \gamma_1 \cdot \gamma_2$

$\gamma_1 : [0, \theta] \ni t \mapsto e^{2\pi i t}$ $\quad (\theta = \arg x)$

$\gamma_2 : [0, -\log|x|] \ni t \mapsto \frac{x}{|x|} e^{-t}$

Estimate on γ_1: by compactness ($\{x=1\} \times \overline{\mathbb{D}}$), $|\tilde{h}_{\gamma_1} \circ \phi \circ h_{\gamma_1^-}(x,y)| \leq M \cdot |y|$

(M constant depending only on $f, \tilde{f}$).

Estimate on γ_2: $\quad |\tilde{h}_{\gamma_2}(x,y)| \leq |y| e^{-\alpha t} e^{|\alpha|(1-e^{-t})/2}$ $\quad t = -\log|x|$

$\quad |h_{\gamma_2^-}(x,y)| \leq |y| e^{\alpha t} e^{|\alpha||x|(e^{t}-1)/2}$ $\Leftarrow$ uses $x \mid f, \tilde{f}$.

All together $\quad |\Phi(x,y)| \leq M e^{|\alpha|(1-|x|)}$. $\square$

Rem: we use strongly the fact that $\alpha < 0$, i.e., the "saddle" behavior, like in the picture. In the node case, the holonomy does not necessarily determine the equivalence class of the foliation.

Siegel foliations with prescribed holonomy

Consider Fol_α := Siegel foliations of index $\alpha < 0$, with marked separatrix $C = \{y=0\}$, up to orbital equivalence.

Consider the map. $hol_C : Fol_\alpha \to \cancel{Aut_\lambda(\mathbb{C},0)} \leftarrow$ germs of the form $z \mapsto \lambda z(1+o(1))$
$\cong$ $\lambda = e^{2\pi i \alpha}$

Mattei-Moussu's result tells that hol_C is injective.

Thm (Martinet-Ramis 1982 ($\alpha \in \mathbb{Q}_{<0}$), Perez-Marco-Yoccoz 1994 ($\alpha \in \mathbb{R}_{<0}$)) hol_C is surjective:
For any $h : (\mathbb{C},0) \circlearrowleft$ such that $h(z) = e^{2\pi i \alpha} z(1+o(1))$, $\exists\, \omega$:
$\omega = x\,dy - \alpha y(1+f(x,y))\,dx$, s.t. the holonomy of the foliation F defined by ω w.r.t. $C = \{y=0\}$ is h.

Rem: Mar-Ram. proof is based on a deep understanding of the conjugacy classes of germs in $Aut_\lambda(\mathbb{C},0)$, following the Ecalle-Voronin classification.
PM-Yoc work without any knowledge of $Aut_\lambda(\mathbb{C},0)$, which is very (unknown) complicated when $\lambda = e^{2\pi i \alpha}$ α not satisfying the Brjuno condition.

Sketch of proof:

Step 1: preparation. Take $\omega = x\,dy - \alpha y(1+f(x,y))\,dx$.
The interesting dynamics happen outside the separatrices: consider $\mathbb{C}^2 \setminus \{xy=0\}$.
Its universal cover is given by
$$E : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{xy=0\}$$
$$(z_1,z_2) \mapsto (e^{2\pi i z_1}, e^{2\pi i z_2})$$

The group of automorphisms of E is generated by the translations $T_1(z)=(z_1+1, z_2)$ and $T_2(z)=(z_1, z_2+1)$.

Notice that a polydisk $\mathbb{D}_r^2 \setminus \{xy=0\}$ is lifted to the area $\mathbb{H}_R^2$, where $\mathbb{H}_R = \{ z \in \mathbb{C} \mid \operatorname{Im} z > R \}$, $R = -\frac{1}{2\pi}\log r$.

We may assume ω defined on $\overline{\mathbb{D}}^2$ $(r=1)$, hence Ω defined on $\overline{\mathbb{H}}^2$ $(R=0)$

The lift of ω to $\mathbb{H}^2$ takes the form $E^*\omega = 2\pi i \cdot e^{2\pi i(z_1+z_2)}\left(dz_2 - \alpha(1+E^*f)\,dz_1\right)$ which is equivalent to $dz_2 - \alpha(1+E^*f)\,dz_1 =: \Omega$

Here E^*f is T_1 and T_2-invariant, and $|E^*f(z_1)| \le C e^{-2\pi \operatorname{Im} z_1 \wedge \operatorname{Im} z_2}$, $z \in \mathbb{H}_R^2$ $R \gg 0$.

We want to control the holonomy of ω w.r.t. $y=0$. For technical reasons, it is easier to work with its inverse (the holonomy of $\gamma(t) = (e^{-2\pi i t}, 0)$)

This holonomy $h(y) = e^{2\pi i|\alpha|}y(1+h_0(y))$ is computed in the transverse $x=1$, and lifts to $H(z_2) = z_2 + |\alpha| + \underbrace{\frac{1}{2\pi i}\log(1+E^*h_0)}_{=:\Phi(z_2)}$

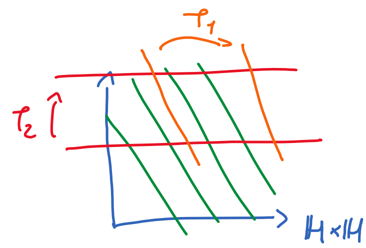
ω and h are recovered from Ω and H, by considering $\mathbb{H}^2 \xrightarrow{E} \mathbb{D}^2 \simeq \mathbb{H}^2/\langle T_1, T_2\rangle$

Notice that the leaves of Ω are very close to the ones of $\Omega_0 := dz_2 - \alpha\, dz_1$ which are the planes $z_2 - \alpha z_1 = \text{const}$.

From here the idea of studying the foliations induced by Ω_0, on $\mathbb{H}^2/\langle \tilde T_1, \tilde T_2\rangle$, with $\tilde T_1, \tilde T_2$ perturbations of T_1 and T_2

Explicitly, we consider $\tilde T_2 = T_2$, and

$\tilde T_1(z_1, z_2) = (z_1+1, z_2 + \Phi(z_2 - \alpha z_1))$



The curve $\gamma_0(t) = (e^{-2\pi i t}, 0)$ lifts to the leaves of ω_0 as $\gamma(t) = (e^{-2\pi i t}, y_0 \cdot e^{2\pi i|\alpha|t})$, which lifts to $\mathbb{H}^2$ to $\Gamma(t) = (-t;\ z_2 + |\alpha| t)$

At $\Gamma(0)$ I have $(0, z_2)$

at $\Gamma(1)$ I have $(-1, z_2+|\alpha|) \sim \tilde T_1(-1, z_2+|\alpha|) = (0, \underbrace{z_2 + |\alpha| + \Phi(z_2)}_{= H(z_2)})$.

Hence the holonomy is the desired one.
What remains to prove is that $\mathbb{H}^2_{\mathbb{R}} / \langle \tilde{\tau}_1, \tilde{\tau}_2 \rangle \simeq \mathbb{D}^2 \setminus \{xy = 0\}$. (or germs)

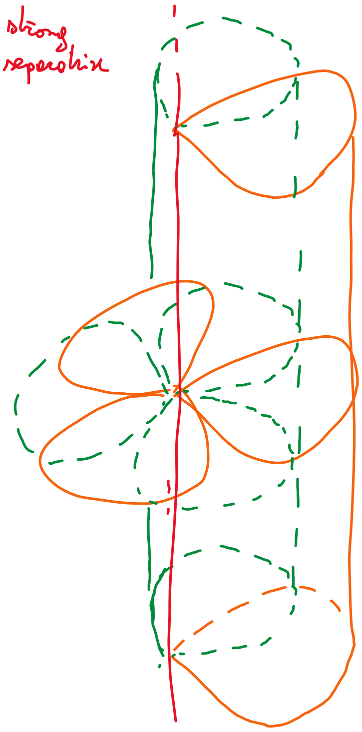
Step 2: C^∞-isomorphism $U := \{ z \mid z_2 + |a| z_1 > 0 \}$ $\tilde{\tau}_1, \tilde{\tau}_2$-invariant.
Build $v : U \to \mathbb{C}^2$ C^∞-diffeomorphism s.t. $v \circ \tilde{\tau}_j = \tau_j \circ v$ $j = 1, 2$
$\Rightarrow M := U / \langle \tilde{\tau}_1, \tilde{\tau}_2 \rangle \overset{C^\infty}{\cong} v(U) / \langle \tau_1, \tau_2 \rangle$
$\Omega_1 := e^{v_1(z)} \cdot \Omega_0$ $v_1 : U \to \mathbb{C}$ C^∞, Ω_1 is $\tilde{\tau}_j$-invariant: $\tilde{\tau}_j^* \Omega_1 = \Omega_1$.

Step 3: Perturbation of v into an analytic diffeomorphism.
- M is a Stein manifold.
- Translation into a $\bar{\partial}$-problem. Use of Hörmander estimates.

Saddle node: sectorial normalization theorem
[Lor. §5.4][Ily-Yak §22I-3][Pym §7]
We recall that a saddle-node is formally orbitally equivalent to the 1-form
$\omega_{r,\beta} = x^{r+1} dy - y(1 + \beta x^r) dx$ for $r \geq 1$ and $\beta \in \mathbb{C}$
It is analytically equivalent to $\omega_{r,\beta} + x^N R_N dx$
The holonomy w.r.t. the strong separatrix $\{x = 0\}$ is formally conjugate to the holonomy of $\omega_{r,\beta}$, which is $h_{r,\beta}(z) = z\left(1 - z^r + \left(\frac{\beta}{2\pi i} + \frac{r+1}{2}\right) z^{2r}\right)$.

Leau-Fatou's theorem describes the dynamics of h quite precisely, by decomposing a punctured neighborhood of the origin into $2r$ attracting and repelling petals (Δ_j^+, Δ_j^-) $j = \mathbb{Z}/r\mathbb{Z}$ where the dynamics is conjugated to $h_{r,\beta}$ (and in fact $h_{r,0}$).
In fact, this picture extends to a neighborhood of 0 (minus $\{x = 0\}$), to describe the foliation using its formal model.

Theorem (sectorial normalization, HUKUHARA-KIMURA-MATUDA, 1961)
There exists attracting and repelling petals (Δ_j^+, Δ_j^-), and fibered analytic diffeomorphisms $\Phi_j^\pm : \Delta_j^\pm \times \mathbb{D}_r \longrightarrow V_j^\pm$, $\Phi_j^\pm(x,y) = (x, \phi_j^\pm(x,y))$
extending continuously as the identity on the strong separatrix $\{0\}\times \mathbb{D}_r$,
which conjugate $\omega|_{\Delta_j^\pm \times \mathbb{D}_r}$ to $\omega_{r,\beta}|_{V_j^\pm}$.



Idea of the proof:
Write the formal conjugacy
$\Phi(x,y) = \left(x, \sum_{n\geq 0} \varphi_n(x)\, y^n\right)$, $\varphi_1 \equiv 1$
Show that φ_n all converge in a uniform sector, plus estimates of growth of $|\varphi_n|$
Technique: resurgence theory (ECALLE, ~ 1981)
uses the conjugacy equation to deduce special convergence properties of Φ (φ_n are 1-Gevrey).

Stokes phenomenon
(behavior of a map depending on the sector)

Rem: in each $\Delta_j^\pm \times \mathbb{D}_r$, $\omega_{r,\beta}$ is orbitally equivalent to $\omega_{r,0}$ via

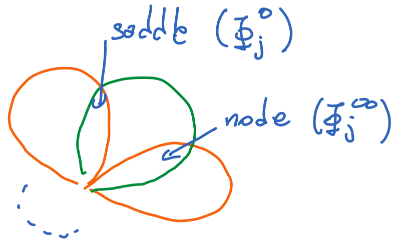
$$(x,y) \mapsto \left(\frac{x}{(1-\beta r x^r \log x)^{\frac{1}{r}}}, y\right)$$

Notice that $\omega_{r,0} = x^{r+1}\, dy - y\, dx$ has the first integral $s = y\, e^{\frac{1}{rx^r}}$

We deduce that on $\Delta_j^\pm \times \mathbb{D}_r$, the foliation looks like $y = c \cdot e^{-\frac{1}{rx^r}}$
In Δ_j^+, $\operatorname{Re}(r\pi i x^r) < 0$, while in Δ_j^-, $\operatorname{Re}(r\pi i x^r) > 0$
attracting repelling

$\Rightarrow$ In $\Delta_j^+ \cap \Delta_j^-$, $\operatorname{Im}(r\pi i x^r) < 0$, i.e., $\operatorname{Re}\left(\frac{-1}{rx^r}\right) < 0$ $\rightarrow$ node behavior
In $\Delta_j^- \cap \Delta_{j+1}^+$, $\operatorname{Im}(r\pi i x^r) > 0$, i.e., $\operatorname{Re}\left(-\frac{1}{rx^r}\right) > 0$ $\rightarrow$ saddle behavior

Analytic classification: related to the fibrated transition maps: $\Phi_j^\infty = \Phi_j^+ \circ \Phi_j^{-\,-1}$ $\quad \Phi_j^0 = \Phi_{j-1}^- \circ \Phi_j^{+\,-1}$



The induced transition maps on the holonomy level are denoted $\varphi_j^\infty, \varphi_j^0$. They determine uniquely the Φ_j^∞, Φ_j^0

$(\varphi_j^\infty, \varphi_j^0)$ is called MARTINET-RAMIS module of ω

It is an ECALLE module for the holonomy h.

Rem: The φ_j^∞ are necessarily affine.

Thm (MARTINET-RAMIS) The MARTINET-RAMIS module (i.e. the holonomy h) determines the equivalence class of ω.

An ECALLE module $(\varphi_j^\infty, \varphi_j^0)$ is realisable as the holonomy of a saddle-node $\iff$ φ_j^∞ is affine $\forall j$.